NORTH-HOLLAND

The Positive Minorant Property on Matrices

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ABSTRACT

We study the positive minorant property for norms on spaces of matrices. A matrix is said to be a majorant of another if all the entries in the first matrix are greater than or equal to the absolute values of the corresponding entries in the second matrix. For a real number $p \geqslant 0$ the Schatten p-norm of the matrix is the l^p-norm of its singular values. The space of $n \times n$ matrices with the Schatten p-norm is said to have the positive minorant property if the norm of each nonnegative matrix is greater than or equal to the norm of every nonnegative matrix that it majorizes. It is easy to show that this property holds if p is even. We show that the positive minorant property fails when $p < 2(n-1)$ and p not even, and provide a simple proof to show the property does hold when $p \geqslant 2(n-1)[(n-1)/2] + 2$. © Elsevier Science Inc., 1997

1. INTRODUCTION

We begin by introducing definitions and reviewing some of the history of the minorant properties on matrices. Our standard reference for notation and terminology related to matrix theory is Horn and Johnson's book [4].

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Let $(M_n(\mathbb{C}), \|\cdot\|_p)$ denote the set of $n \times n$ matrices with complex entries, equipped with the Schatten p-norm $\|\cdot\|_p$. If $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A) \geqslant 0$ denote the singular values of $A \in M_n(\mathbb{C})$, then for $1 \leqslant p < \infty$, the Schatten p-norm of A is defined by

$$\|A\|_p = \left(\sum_{i=1}^{n} \sigma_i(A)^p \right)^{1/p} = \left(\operatorname{tr}\left[(A^*A)^{p/2}\right]\right)^{1/p}.$$

While $\|\cdot\|_p$ is not a norm for $0 < p < 1$, it is still well defined and will be considered. When $1 \leqslant p < \infty$, it is well known that $(M_n(\mathbb{C}), \|\cdot\|_p)$ is a Banach space [5, 10].

In this paper, for $A \in M_n(\mathbb{C})$, we will define $|A|$ to be the entrywise absolute value of A, that is, the matrix $(|a_{ij}|)$. For $A, B \in M_n(\mathbb{R})$, write $A \leqslant B$ or $B \geqslant A$ if $a_{ij} \leqslant b_{ij}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. In particular, $A \geqslant 0$ means that all the entries of A are nonnegative.

Definition 1.1. If $A, B \in M_n(\mathbb{C})$ with $|A| \leqslant B$, then we say B is a *majorant* of A (or that B *majorizes* A).

Definition 1.2. The space $(M_n(\mathbb{C}), \|\cdot\|_p)$ is said to have the *positive minorant property* if, for $A, B \in M_n(\mathbb{C})$, $\|A\|_p \leqslant \|B\|_p$ whenever $0 \leqslant A \leqslant B$.

The positive minorant property was first introduced by Déchamps-Gondim, Lust-Piquard, and Queffelec [2]. It is a weaker formulation of the minorant property, a property first introduced by Hardy and Littlewood [3] in the context of Fourier analysis on the spaces $L^p(\mathbb{T})$. In the setting of matrices, the minorant property can be expressed as follows.

Definition 1.3. The space $(M_n(\mathbb{C}), \|\cdot\|_p)$ is said to have the *minorant property* if, for $A, B \in M_n(\mathbb{C})$, $\|A\|_p \leqslant \|B\|_p$ whenever B is a majorant of A (that is, whenever $|A| \leqslant B$).

It is not known whether the positive minorant property is equivalent to the minorant property. It is worth noting, however, that if we define the properties for subspaces, there exists examples where the properties are not equivalent. For example [11], the space of 2×2 circulant matrices has the positive minorant property but fails to have the minorant property for $1 \leqslant p < 2$.

It is easy to prove that the minorant property (and hence also the positive minorant property) holds when p is an even positive integer. That is, if $p = 2k$, where $k \in \mathbb{N}$, and if $|A| \leqslant B$, then it is trivial to show that $|A^*A| \leqslant B^*B$ and consequently $|(A^*A)^k| \leqslant (B^*B)^k$, so it follows that $\|A\|_p^p \leqslant \|B\|_p^p$.

However, the minorant properties do not hold in general. In this paper we show that the space $(M_n(\mathbb{C}), \|\cdot\|_p)$ fails to have either property whenever $p < 2(n-1)$ and p not an even integer. As a consquence, it is elementary to prove Peller's [7] result that the minorant property holds on the Banach space of compact operators on l^2 with finite Schatten p-norm only if p is an even positive integer. Moreover, we can now extend this result to the positive minorant property.

Rosen [8] has shown that the positive minorant property holds whenever $p > 2(n-1)$. Combining our results with Rosen's yields the complete answer to the question as to when the positive minorant property holds: the space $(M_n(\mathbb{C}), \|\cdot\|_p)$ has the positive minorant property if and only if $p > 2(n-1)$ or p is an even integer. It appears that the question of when the minorant property holds has been completely answered only for 2×2 and 3×3 matrices [11, 2]. The space $(M_2(\mathbb{C}), \|\cdot\|_p)$ has the minorant property if and only if $p \geqslant 2$. The space $(M_3(\mathbb{C}), \|\cdot\|_p)$ has the minorant property if and only if $p \geqslant 4$ or $p = 2$.

That the minorant property fails on the spaces $(M_n(\mathbb{C}), \|\cdot\|_p)$ was first shown by Simon [9] using matrices based on a counterexample of Boas [1], who was studying the minorant property on L^p spaces. Simon proved that if p is not an even integer, then the minorant property fails for $n = 2[p/2] + 5$. As any counterexample for $n \times n$ matrices can be embedded in matrices of a higher dimension, for a fixed p Simon considered the smallest n for which the minorant property fails, which he denoted $N(p)$. Accordingly, the principal result of [9] asserts that

$$N(p) \leqslant 2\left[\frac{p}{2}\right] + 5.$$

Déchamps-Gondim, Lust-Piquard, and Queffelec [2] improved on this by showing

$$N(p) \leqslant \left[\frac{p}{2}\right] + 2,$$

or that $(M_n(\mathbb{C}), \|\cdot\|_p)$ fails to have the minorant property for $1 \leqslant p < 2(n-1)$ and p noneven. They further conjecture that $(M_n(\mathbb{C}), \|\cdot\|_p)$ has the minorant property if $p \geqslant 2(n-1)$, but this does not seem to have been verified beyond 3×3 matrices. A positive solution to this conjecture along with our present result for the failure of the positive minorant property would show that the minorant and positive minorant properties on the space $(M_n(\mathbb{C}), \|\cdot\|_p)$ were indeed equivalent, and would hold if and only if p is even or $p \geqslant 2(n-1)$. Earlier this would not have been such a reasonable conjecture. Until now the best results for the failure of the positive minorant property were given by the authors of [2], namely: For n even and $n \geqslant 4$, the space $(M_n(\mathbb{C}), \|\cdot\|_p)$ does not have the positive minorant property if $1 \leqslant p < n-2$ and p noneven. For n odd and $n \geqslant 5$, the space $(M_n(\mathbb{C}), \|\cdot\|_p)$ does not have the positive minorant property if $1 \leqslant p < n-3$ and p noneven.

The conjecture that the minorant property holds if $p \geqslant 2(n-1)$ does not appear to be easy to prove. However, our methods show that $(M_n(\mathbb{C}), \|\cdot\|_p)$ has the positive minorant property if $p \geqslant 2(n-1)[(n-1)/2]+2$. At about the same time this was done, Rosen [8] using less elementary methods was able to close the gap and show that the positive minorant property holds for $p \geqslant 2(n-1)$.

The interest in the Schatten p-norms and precisely the difficulty in dealing with them is that they do not respect the natural ordering on $M_n(\mathbb{C})$ induced by the cone of matrices with nonnegative entries. On the other hand, they are the most natural norms to consider and arise in the context of similar problems studied from the point of view of Fourier analysis. In this context, given a compact abelian group G equipped with the normalized Haar measure and functions $f, g \in L^p(G)$ such that $|\hat{f}(\gamma)| \leqslant \hat{g}(\gamma)$ for every γ in $\hat{G}$, it is natural to ask whether $\|f\|_p \leqslant \|g\|_p$. In [11] it is shown that when G is a finite group of order n, the space $L^p(G)$ can be embedded in $(M_n(\mathbb{C}), \|\cdot\|_p)$. Under this embedding the Fourier coefficients of the functions become matrix entries and the L^p-norm is equivalent to the Schatten p-norm.

The minorant problem becomes trivial for many of the widely used matrix norms. It is not too difficult to show the minorant properties will hold for any matrix norm induced by a monotone vector norm. Recall that the matrix norm *induced* by the vector norm $\|\cdot\|$ on $\mathbb{C}^n$ is defined as

$$|||A||| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

A vector norm is said to be *monotone* if $\|\mathbf{x}\| \leqslant \|\mathbf{y}\|$ whenever $|\mathbf{x}| \leqslant |\mathbf{y}|$. Given matrices $A, B \in M_n(\mathbb{C})$ such that $|A| \leqslant B$ (or $0 \leqslant A \leqslant B$ when considering the positive minorant property), note that for $\mathbf{x} \in \mathbb{C}^n$ we have that $|A\mathbf{x}| \leqslant$

$B|\mathbf{x}|$. Thus $\|A\mathbf{x}\| \leqslant \|B|\mathbf{x}|\,\|$ as the vector norm is monotone. Hence we have

$$\begin{aligned}
|||\,A\,||| &= \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \\
&\leqslant \max_{\|\mathbf{x}\|=1} \|\,B|\mathbf{x}|\,\| \\
&\leqslant \max_{\|\mathbf{x}\|=1} \|B\mathbf{x}\| \\
&= |||\,B\,|||,
\end{aligned}$$

and thus the (positive) minorant property holds. It now follows that the matrix p-norms $|||\cdot|||_p$ induced by l_p satisfy the minorant properties. One should not confuse the matrix norms induced by the l_p vector norms ($p \geqslant 1$) with the Schatten p-norms. The former matrix norms have the minorant properties for all $p \geqslant 1$.

2. FAILURE OF THE POSITIVE MINORANT PROPERTY

THEOREM 2.1. *The space* $(M_n(\mathbb{R}), \|\cdot\|_p)$ *does not have the positive minorant property for* $0 < p < 2(n-1)$ *and* p *not an even integer.*

To prove this theorem when $p > 1$, we will use the following theorem.

THEOREM 2.2. *For* $1 < p < \infty$, *the space* $(M_n(\mathbb{R}), \|\cdot\|_p)$ *has the positive minorant property if and only if* $(B^*B)^{(p/2)-1}B^* \geqslant 0$ *whenever* $B \geqslant 0$.

This theorem is proved in [2], appearing as Theorem 3(1). The authors of [2] explain what they mean by $(B^*B)^{(p/2)-1}B^*$ when $1 < p < 2$ and B is not invertible. To prove Theorem 2.1 for $p > 1$ it suffices for us to find a nonnegative invertible matrix B for which $(B^*B)^{(p/2)-1}B^*$ has a negative entry. Specifically, we show this for the matrix

$$A = E_{11} + J_n + J_n^T,$$

where E_{11} is the matrix with all entries zero except for a single 1 in the upper left hand corner, and J_n is the Jordan matrix with all entries zero except for

1's just above the main diagonal. That is,

$$A = \begin{pmatrix} 1 & 1 & 0 & & & & \\ 1 & 0 & 1 & & & \mathbf{0} & \\ 0 & 1 & 0 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & 0 & 1 & 0 \\ & \mathbf{0} & & & & 1 & 0 & 1 \\ & & & & & 0 & 1 & 0 \end{pmatrix}.$$

This matrix has the property that the last entry in $(A^*A)^{(p/2)-1}A^* = A^{p-1}$ is negative for $2(n-2) < p < 2(n-1)$. To deal with smaller values of p we build other nonnegative matrices which we denote by B_k and which have the property that $(B_k^* B_k)^{(p/2)-1}B_k^*$ has a negative last entry for $2(k-2) < p < 2(k-1)$ for $k = 2, 3, \dots, n-1$. Let A_r denote the leading $r \times r$ principal submatrix of A, and let B_k be the matrix with blocks

$$\begin{pmatrix} I_{n-k} & 0 \\ 0 & A_k \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix. Then $B_k = B_k^* \in M_n(\mathbb{R})$ and

$$(B_k^* B_k)^{(p/2)-1} B_k^* = B_k^{p-1} = \begin{pmatrix} I_{n-k} & 0 \\ 0 & (A_k^* A_k)^{(p/2)-1} A_k^* \end{pmatrix}.$$

Our analysis of the full matrix $A_n = A$ will show that for $k \geqslant 2$,

$$\left[(A_k^* A_k)^{(p/2)-1} A_k^* \right]_{kk} < 0 \qquad \text{for} \quad 2(k-2) < p < 2(k-1),$$

so that $[(B_k^* B_k)^{(p/2)-1} B_k^*]_{nn} < 0$ when $2(k-2) < p < 2(k-1)$, for $k = 2, 3, \dots, n$.

It then follows from the proof of Theorem 2.2 that for small enough positive values of ε the matrix E_k obtained from B_k by adding ε in the last entry has a strictly smaller p-norm than B_k does, although E_k majorizes B_k.

We cannot use Theorem 2.2 when $0 < p \leqslant 1$. Instead we explicitly provide a counterexample. Let

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since U and V are symmetric matrices, their singular values are just the absolute values of their eigenvalues, which a simple calculation shows to be $(1 \pm \sqrt{5})/2$ for U and 0, 2 for V. Hence,

$$\|U\|_p = \left[\left(\frac{\sqrt{5} - 1}{2} \right)^p + \left(\frac{\sqrt{5} + 1}{2} \right)^p \right]^{1/p} \geqslant \sqrt{5}$$

if $0 < p \leqslant 1$ by Jensen's inequality, while

$$\|V\|_p = (0^p + 2^p)^{1/p} = 2,$$

and so we have that $\|V\|_p < \|U\|_p$ for all $0 < p \leqslant 1$. As V majorizes U, we have shown that the positive minorant property fails on the space $(M_2(\mathbb{R}), \|\cdot\|_p)$ for all $0 < p \leqslant 1$. For the spaces $(M_n(\mathbb{R}), \|\cdot\|_p)$ we simply consider two $n \times n$ matrices whose entries are all zero except that one has a copy of U and the other of V in the top left corner. The "norms" of these matrices are the same as the corresponding "norm" of U or V, providing the needed counterexample.

The case when $p > 1$ is more difficult due to the delicate relationship between the size of the matrices and value of p. Larger values of p require larger matrices to cause the failure of the minorant properties.

2.1. *Eigenvalues and Eigenvectors of A*

Before we can proceed with the proof of Theorem 2.1 for the case when $p > 1$, we need to know a few facts about the eigenvalues and eigenvectors of A. We will also have to express the last entry of $(A^*A)^{(p/2)-1}A^*$ in terms of these eigenvalues and eigenvectors.

To begin with, A is a symmetric matrix, so its eigenvalues are real, and there is an orthonormal basis of eigenvectors of A. We will show that none of the eigenvalues is 0 and that they are distinct.

List the eigenvalues as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Letting $\mathbf{x}_j$ denote an eigenvector associated with the eigenvalue λ_j, then $A\mathbf{x}_j = \lambda_j \mathbf{x}_j$ for $j = 1, \ldots, n$, or by equating each row

$$x_{1j} + x_{2j} = \lambda_j x_{1j}, \tag{1}$$

$$x_{i-1\,j} + x_{i+1\,j} = \lambda_j x_{ij}, \qquad i = 2, \dots, n-1, \tag{2}$$

$$x_{n-1\,j} = \lambda_j x_{nj}. \tag{3}$$

If $\lambda_j = 0$, Equation (3) says $x_{n-1\,j} = 0$, but then Equation (2) with $i = n-2$ says $x_{n-3j} + x_{n-1j} = 0$, so $x_{n-3\,j} = 0$, which in turn says $x_{n-5j} = 0$, etc., until we get either $x_{1j} = 0$ or $x_{2j} = 0$. Then Equation (1) says $x_{1j} + x_{2j} = 0$, so both x_{1j} and $x_{2j} = 0$, and so, using the equations (2), we can conclude $\mathbf{x}_j = 0$. However, $\mathbf{x}_j$ is an eigenvector, so we have a contradiction, and none of the eigenvalues of A can be zero.

Turning our attention now to the eigenvectors of A, we suppose $x_{nj} = 0$. Then Equation (3) says $x_{n-1\,j} = 0$, and then the equations (2), working backwards, give in turn that $x_{n-2\,j} = 0$, $x_{n-3\,j} = 0, \dots, x_{1j} = 0$. Again our supposition has produced a contradiction, so $x_{nj} \neq 0$.

Putting $x_{nj} = 1$, then by (3) $x_{n-1\,j} = \lambda_j$, and by (2)

$$x_{i-1\,j} = \lambda_j x_{i\,j} - x_{i+1\,j} \qquad \text{for} \quad i = 2, \dots, n-1.$$

Hence we can solve for $\mathbf{x}_j$ using this recursive method. It shows that each eigenspace is one-dimensional and the eigenvalues are distinct. Equation (1) turns out to be redundant when λ_j is an eigenvalue.

Let $\mathbf{p}_j = \mathbf{x}_j/\|\mathbf{x}_j\|_2$ be normalized eigenvectors of A, and put $P = (\mathbf{p}_1, \dots, \mathbf{p}_n) = (p_{i\,j})$. Letting $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then

$$AP = P\Lambda.$$

Since A is symmetric, P is an orthonormal matrix. Now $A^*A = A^2 = P\Lambda^2 P^T$, so

$$\begin{aligned}(A^*A)^{(p/2)-1} &= P(\Lambda^2)^{(p/2)-1} P^T \\ &= P|\Lambda|^{p-2} P^T\end{aligned}$$

where $|\Lambda| = \operatorname{diag}(|\lambda_1|, \dots, |\lambda_n|)$.

Thus

$$\begin{aligned}\left[(A^*A)^{(p/2)-1}A^*\right]_{nn} &= \left[P|\Lambda|^{p-2}\Lambda P^T\right]_{nn}\\ &= \sum_{j=1}^{n} p_{nj}\left[|\Lambda|^{p-2}\Lambda P^T\right]_{jn}\\ &= \sum_{j=1}^{n} p_{nj}|\lambda_j|^{p-2}\lambda_j p_{jn}^T\\ &= \sum_{j=1}^{n} p_{nj}^2 \operatorname{sgn}\lambda_j\,|\lambda_j|^{p-1}.\end{aligned}$$

We define

$$f(p) = \left[(A^*A)^{(p/2)-1}A^*\right]_{nn} = \sum_{j=1}^{n} p_{nj}^2 \operatorname{sgn}\lambda_j\,|\lambda_j|^{p-1}, \tag{4}$$

and as noted earlier $\lambda_j \neq 0$, so f is well defined for all real numbers p.

2.2. *When the Last Entry is Zero*

We now consider $(A^*A)^{(p/2)-1}A^*$ and will show that the last entry of this matrix is negative for $2(n-2) < p < 2(n-1)$, that is,

$$f(p) = \left[(A^*A)^{(p/2)-1}A^*\right]_{nn} < 0.$$

To do this we first show that this entry is 0 for certain even integer values of p.

LEMMA 2.3. *The quantity $f(p) = 0$ if $p = 2k$, for $k = 1, 2, \dots, n-1$.*

Proof. Notice that

$$\begin{aligned}f(2k) &= \left[(A^*A)^{k-1}A^*\right]_{nn}\\ &= \left[A^{2k-1}\right]_{nn} \qquad \text{(as } A \text{ is real and symmetric)}\\ &= \mathbf{e}_n^T A^{2k-1}\mathbf{e}_n,\end{aligned}$$

where $\mathbf{e}_n$ is the n-dimensional unit vector $(0, \dots, 0, 1)^T$. Write $A = E_{11} + J_n + J_n^T$, where we think of J_n as the shift-up operator and J_n^T as the shift-down operator. That is, for a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$ we have that $J_n\mathbf{v} = (v_2, v_3, \dots, v_n, 0)^T$ and $J_n^T\mathbf{v} = (0, v_1, \dots, v_{n-1})^T$. Now consider the terms in the expansion of

$$A^{2k-1} = \left(E_{11} + J_n + J_n^T\right)^{2k-1}. \tag{5}$$

Let T denote one such term, that is,

$$T = \prod_{i=1}^{2k-1} X_i$$

where $X_i = E_{11}$, J_n, or J_n^T for each $i = 1, \dots, 2k-1$.

If T has no factor of E_{11}, then T is the product of an odd number of shift operators. This means that $T\mathbf{e}_n$ can never be $\mathbf{e}_n$, and hence $\mathbf{e}_n^T T\mathbf{e}_n = 0$.

On the other hand, if T has at least one factor of E_{11}, then unless there are at least $n-1$ shift operators following the last appearance of E_{11} in T, $T\mathbf{e}_n$ will be the zero vector. That is, for any vector $\mathbf{v}$, $E_{11}\mathbf{v} = (v_1, 0, \dots, 0)^T$, so that in particular if $\mathbf{v}$ is a vector obtained only by shift operators acting on the vector $\mathbf{e}_n$, we would need at least $n-1$ shift-up operators before it is possible that $E_{11}\mathbf{v} \neq 0$. Similarly, $\mathbf{e}_n^T T = 0$ unless there are at least $n-1$ shift-down operators preceding the first appearance of E_{11} in T. Hence if T has less than $2(n-1) + 1 = 2n - 1$ factors, it follows that $\mathbf{e}_n^T T\mathbf{e}_n = 0$.

Actually, T has $2k - 1$ factors, so in either situation, when $k < n$, we get that $\mathbf{e}_n^T T\mathbf{e}_n = 0$, and since T is an arbitrary term from the expansion of (5), we obtain that $\mathbf{e}_n^T A^{2k-1}\mathbf{e}_n = 0$. That is, $f(2k) = 0$ for $k = 1, 2, \dots, n-1$. ■

2.3. *When the Last Entry Is Not Zero*

We will show that the zeros specified in Lemma 2.3 are the only zeros of f. We use the following lemma, whose proof can be found in [6].

LEMMA 2.4. *Given two sequences of real constants $\{t_j\}$ and $\{c_j\}$ with constants t_j positive and distinct and not all $c_j = 0$, define the function $h(\alpha) = \sum_{j=1}^{n} c_j t_j^{\alpha}$. Then, counting multiplicity, h has at most $n-1$ zeros.*

We need to check that $f(p) = \sum_{j=1}^{n} p_{nj}^2 \operatorname{sgn} \lambda_j |\lambda_j|^{p-1}$ given in (4) is of the form $\sum_{j=1}^{n} c_j t_j^{\alpha}$ specified in Lemma 2.4. Let $\alpha = p - 1$, and $c_j = p_{nj}^2 \operatorname{sgn} \lambda_j$ and $t_j = |\lambda_j|$ for all j. Observe that the individual coefficients c_j cannot be 0, because $p_{nj} = 1/\|\mathbf{x}_j\|_2$ and no eigenvalue is 0.

While the eigenvalues are distinct, it may appear possible that their absolute values might not be distinct. If $|\lambda_j| = |\lambda_{j'}|$ for distinct j and j', then combining the corresponding terms in the sum for $f(p)$ yields a similar expression with fewer terms. We need to show that this expression does not reduce to the zero function. Since the trace of the matrix A is 1, the list of eigenvalues cannot just consist of pairs with the same absolute value and opposite signs. So f cannot be identically equal to 0.

Applying Lemma 2.4 to f shows that it has at most $n - 1$ zeros. Since we found $n - 1$ distinct zeros in Lemma 2.3, they must all be simple. It follows that the sign of f changes at each of its zeros.

The space $(M_n(\mathbb{R}), \|\cdot\|_p)$ has the positive minorant property when $p = 2k$, so that $(B^*B)^{(p/2)-1}B^* \geqslant 0$ for matrices $B \geqslant 0$, and in particular, $f(p) = [(A^*A)^{(p/2)-1}A^*]_{nn} \geqslant 0$ when $p = 2k$. More precisely, since $f(2k) = 0$ when $k = 1, \dots, n-1$, we must have $f(2k) > 0$ for all integers $k \geqslant n$. It follows that $f(p) > 0$ for all $p > 2(n-1)$, since it has no zeros in this interval and is positive at some points in the interval. So the sign change of f at $p = 2(n-1)$ must go from negative to positive. Hence $f(p) < 0$ for all $p \in \,]2(n-2), 2(n-1)[$. This completes the proof of Theorem 2.1.

3. WHEN THE POSITIVE MINORANT PROPERTY HOLDS

THEOREM 3.1. *The space $(M_n(\mathbb{R}), \|\cdot\|_p)$ has the positive minorant property for*

$$p \geqslant 2(n-1)\left[\frac{n-1}{2}\right] + 2.$$

Proof. It will be enough to show that $(B^*B)^{(p/2)-1} \geqslant 0$ whenever $B \in M_n(\mathbb{R})$ with $B \geqslant 0$, as it then trivially follows that $(B^*B)^{(p/2)-1}B^* \geqslant 0$ and hence the space $(M_n(\mathbb{R}), \|\cdot\|_p)$ will have the positive minorant property by Theorem 2.2.

We argue by contradiction. Suppose there exists a matrix $B \in M_n(\mathbb{R})$ such that $B \geqslant 0$ but for which $(B^*B)^{(p/2)-1}$ has a negative entry for some $p > 2(n-1)[(n-1)/2] + 2$. That is, suppose that

$$\left[(B^*B)^{(p/2)-1}\right]_{ij} < 0.$$

As in the proof of the last theorem, we shall express the entries of $(B^*B)^{(p/2)-1}$ in terms of the eigenvalues and eigenvectors of B^*B, which will be functions of the form described in Lemma 2.4. This being the case, each entry of $(B^*B)^{(p/2)-1}$ considered as a function of p can have at most $n-1$ zeros, however, we show that our supposition forces at least one of the entries in the jth column to have more than $n-1$ zeros, producing a contradiction.

Let $b_{kl}(\alpha)$ denote the row$k$, column$l$ entry of $(B^*B)^{\alpha}$, which is well defined provided $\alpha > 0$. The matrix B^*B is positive definite and diagonalizable. So letting $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ denote the eigenvalues of B^*B and letting Q denote the matrix of normalized eigenvectors, so that $B^*B = Q\Sigma Q^T$, we find that

$$\begin{aligned} b_{kl}(\alpha) &= \left[(B^*B)^{\alpha}\right]_{kl} \\ &= \left[(Q\Sigma Q^T)^{\alpha}\right]_{kl} \\ &= \left[Q\Sigma^{\alpha}Q^T\right]_{kl} \\ &= \sum_{m=1}^{n} q_{km}q_{lm}\sigma_m^{\alpha}. \end{aligned} \tag{6}$$

We also note that

$$\begin{aligned} b_{kl}(\alpha) &= \left[(B^*B)^{\alpha}\right]_{kl} \\ &= \left[(B^*B)(B^*B)^{\alpha-1}\right]_{kl} \qquad (\text{provided } \alpha > 1) \\ &= \sum_{m=1}^{n} \left[(B^*B)\right]_{km}\left[(B^*B)^{\alpha-1}\right]_{ml} \\ &= \sum_{m=1}^{n} \left[(B^*B)\right]_{km} b_{ml}(\alpha - 1). \end{aligned} \tag{7}$$

If we know that $b_{kl}(\alpha) < 0$, then by (7) and since $B^*B \geqslant 0$, we must have at least one m such that $b_{ml}(\alpha - 1) < 0$.

Letting $i_1 = i$, our supposition that

$$b_{ij}\left(\frac{p}{2} - 1\right) = \left[(B^*B)^{(p/2)-1}\right]_{ij} < 0$$

and our last remark allow us to define successively finitely many indices i_k, $k = 1, \dots, [p/2]$, such that $b_{i_k j}(p/2 - k) < 0$. We are assuming that $p > 2(n-1)[(n-1)/2] + 2$, so k may be taken to be as large as $(n-1)[(n-1)/2] + 1$. As (6) tells us that $b_{jj}(\alpha) = \sum_{m=1}^{n} q_{jm}^2 \sigma_m^{\alpha} \geqslant 0$, it follows that i_k is different from j for all k. Consequently there are $(n-1)[(n-1)/2] + 1$ points for which, at each point, at least one of $n-1$ functions or entries from the jth column of $(B^*B)^{\alpha}$ is negative. Consequently it is impossible for all these $n-1$ functions to be negative at fewer than $[(n-1)/2] + 1$ of the given points.

Let $b_{lj}(\alpha)$, denote a function from the jth column which is negative for at least $[(n-1)/2] + 1$ of the points $p/2 - k$, $k = 1, \dots, (n-1)[(n-1)/2] + 1$. As p is noneven, none of the points $p/2 - k$ is an integer. However, when r is even the space $(M_n(\mathbb{R}), \|\cdot\|_r)$ has the positive minorant property (see Introduction). It follows from Theorem 2.2 that $(B^*B)^{(r/2)-1}B^* \geqslant 0$ and consequently $(B^*B)^{r/2} \geqslant 0$, that is, $b_{lj}(\alpha) \geqslant 0$ whenever $\alpha \in \mathbb{N}$. Hence by the intermediate-value theorem, for each k such that $b_{lj}(p/2 - k) < 0$ there are at least two zeros in the closed interval bounded by $[p/2] - k$ and $[p/2] - k + 1$. In total we get at least $2[(n-1)/2] + 2$ zeros counting multiplicity. However, $2[(n-1)/2] + 2 \geqslant n$, and Lemma 2.4 says that functions of the form (7) can have at most $n-1$ zeros counting multiplicity, a contradiction. ∎

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